Exchange economy with Cobb-Douglas utility functions

In an economy there are two consumers, A and B, who consume two goods, good x and good y. The preferences of these consumers are given by the following utility functions:

$$u_A = (x_A \cdot y_A)^2$$

$$u_B = \ln(x_B) + 2\ln(y_B)$$

Consumer A has an initial endowment $\omega_A = (9, 2)$, while consumer B has an initial endowment $\omega_B = \left(\frac{3}{2}, 3\right)$. Since these consumers spend all their income and take prices as given, answer the following:

1. What relative price equilibrates the market? What are the consumption bundles of both consumers after trade?
2. What conditions must hold to ensure that we are in a market equilibrium (Walrasian equilibrium)?
3. Find the contract curve and the Utility Possibility Frontier (UPF)
4. Verify that the equilibrium found in part 1) is efficient, using your result from part 3)

Solution

1. Let prices be (p_x, p_y)
   Consumer A solves

$$\max_{x_A, y_A} \; (x_A y_A)^2$$

subject to

$$p_x x_A + p_y y_A = p_x \cdot 9 + p_y \cdot 2$$

Since $(x_A y_A)^2$ is a monotonic transformation of $x_A y_A$, consumer A solves the equivalent problem

$$\max_{x_A, y_A} \; x_A y_A$$

subject to

$$p_x x_A + p_y y_A = 9p_x + 2p_y$$

The Lagrangian is

$$\mathcal{L}_A = x_A y_A + \lambda_A \left(9p_x + 2p_y - p_x x_A - p_y y_A\right)$$

The first-order conditions are

$$\frac{\partial \mathcal{L}_A}{\partial x_A} = y_A - \lambda_A p_x = 0$$

$$\frac{\partial \mathcal{L}_A}{\partial y_A} = x_A - \lambda_A p_y = 0$$

$$\frac{\partial \mathcal{L}_A}{\partial \lambda_A} = 9p_x + 2p_y - p_x x_A - p_y y_A = 0$$

From the first two conditions,

$$\frac{y_A}{p_x} = \frac{x_A}{p_y}$$

$$p_y y_A = p_x x_A$$

Thus consumer A spends one half of income on each good, so

$$x_A = \frac{9p_x + 2p_y}{2p_x}$$

$$y_A = \frac{9p_x + 2p_y}{2p_y}$$

Now consumer B solves

$$\max_{x_B, y_B} \; \ln(x_B) + 2\ln(y_B)$$

subject to

$$p_x x_B + p_y y_B = p_x \cdot \frac{3}{2} + p_y \cdot 3$$

The Lagrangian is

$$\mathcal{L}_B = \ln(x_B) + 2\ln(y_B) + \lambda_B \left(\frac{3}{2}p_x + 3p_y - p_x x_B - p_y y_B \right)$$

The first-order conditions are

$$\frac{\partial \mathcal{L}_B}{\partial x_B} = \frac{1}{x_B} - \lambda_B p_x = 0$$

$$\frac{\partial \mathcal{L}_B}{\partial y_B} = \frac{2}{y_B} - \lambda_B p_y = 0$$

$$\frac{\partial \mathcal{L}_B}{\partial \lambda_B} = \frac{3}{2}p_x + 3p_y - p_x x_B - p_y y_B = 0$$

From the first two conditions,

$$\frac{1/x_B}{2/y_B} = \frac{p_x}{p_y}$$

$$\frac{y_B}{2x_B} = \frac{p_x}{p_y}$$

$$p_y y_B = 2p_x x_B$$

Thus consumer B spends one third of income on good x and two thirds on good y, so

$$x_B = \frac{\frac{3}{2}p_x + 3p_y}{3p_x}$$

$$y_B = \frac{2\left(\frac{3}{2}p_x + 3p_y\right)}{3p_y}$$

Now impose market clearing. Total endowments are

$$\bar{x} = 9 + \frac{3}{2} = \frac{21}{2}$$

$$\bar{y} = 2 + 3 = 5$$

Using market clearing in good y,

$$y_A + y_B = 5$$

$$\frac{9p_x + 2p_y}{2p_y} + \frac{2\left(\frac{3}{2}p_x + 3p_y\right)}{3p_y} = 5$$

$$\frac{9p_x + 2p_y}{2p_y} + \frac{p_x + 2p_y}{p_y} = 5$$

Multiplying by $2p_y$,

$$9p_x + 2p_y + 2p_x + 4p_y = 10p_y$$

$$11p_x + 6p_y = 10p_y$$

$$11p_x = 4p_y$$

$$\frac{p_x}{p_y} = \frac{4}{11}$$

We now substitute this relative price into the demands

For consumer A,

$$x_A = \frac{9p_x + 2p_y}{2p_x} = \frac{9 \cdot \frac{4}{11} + 2}{2 \cdot \frac{4}{11}} = \frac{\frac{36}{11} + \frac{22}{11}}{\frac{8}{11}} = \frac{58}{8} = \frac{29}{4}$$

$$y_A = \frac{9p_x + 2p_y}{2p_y} = \frac{9 \cdot \frac{4}{11} + 2}{2} = \frac{\frac{36}{11} + \frac{22}{11}}{2} = \frac{58}{22} = \frac{29}{11}$$

For consumer B,

$$x_B = \frac{\frac{3}{2}p_x + 3p_y}{3p_x} = \frac{\frac{3}{2} \cdot \frac{4}{11} + 3}{3 \cdot \frac{4}{11}} = \frac{\frac{6}{11} + \frac{33}{11}}{\frac{12}{11}} = \frac{39}{12} = \frac{13}{4}$$

$$y_B = \frac{2\left(\frac{3}{2}p_x + 3p_y\right)}{3p_y} = \frac{2\left(\frac{6}{11} + \frac{33}{11}\right)}{3} = \frac{2 \cdot \frac{39}{11}}{3} = \frac{26}{11}$$

Therefore, the equilibrium allocation is

$$(x_A, y_A) = \left(\frac{29}{4}, \frac{29}{11}\right)$$

$$(x_B, y_B) = \left(\frac{13}{4}, \frac{26}{11}\right)$$

Hence, the market-clearing relative price is $\frac{p_x}{p_y} = \frac{4}{11}$, and the equilibrium consumption bundles are $\left(\frac{29}{4}, \frac{29}{11}\right)$ for consumer A and $\left(\frac{13}{4}, \frac{26}{11}\right)$ for consumer B

2. A Walrasian equilibrium in this exchange economy is a price vector (p_x, p_y) with $p_x > 0$ and $p_y > 0$, together with an allocation

$$(x_A, y_A), (x_B, y_B)$$

such that the following conditions hold

(a) **Utility maximization**

Given prices, each consumer chooses the most preferred bundle in their budget set

Consumer A solves

$$(x_A, y_A) \in \arg \max_{x_A \geq 0,\ y_A \geq 0} \left\{(x_A y_A)^2 \ : \ p_x x_A + p_y y_A \leq p_x \cdot 9 + p_y \cdot 2\right\}$$

Consumer B solves

$$(x_B, y_B) \in \arg \max_{x_B > 0,\ y_B > 0} \left\{\ln(x_B) + 2\ln(y_B) \ : \ p_x x_B + p_y y_B \leq p_x \cdot \frac{3}{2} + p_y \cdot 3\right\}$$

Since both utility functions are strictly increasing in both goods, the budget constraints hold with equality at equilibrium

$$p_x x_A + p_y y_A = p_x \cdot 9 + p_y \cdot 2$$

$$p_x x_B + p_y y_B = p_x \cdot \frac{3}{2} + p_y \cdot 3$$

(b) **Feasibility and market clearing**

Total consumption must equal total endowments in both markets

$$x_A + x_B = 9 + \frac{3}{2} = \frac{21}{2}$$

$$y_A + y_B = 2 + 3 = 5$$

(c) **Positive prices**

Prices must satisfy

$$p_x > 0 \qquad p_y > 0$$

Using the result from part (1), we can quickly verify these conditions

The candidate equilibrium was

$$\frac{p_x}{p_y} = \frac{4}{11}$$

$$(x_A, y_A) = \left(\frac{29}{4}, \frac{29}{11}\right) \qquad (x_B, y_B) = \left(\frac{13}{4}, \frac{26}{11}\right)$$

Feasibility holds because

$$\frac{29}{4} + \frac{13}{4} = \frac{42}{4} = \frac{21}{2}$$

$$\frac{29}{11} + \frac{26}{11} = \frac{55}{11} = 5$$

If we normalize $p_y = 11$, then $p_x = 4$, so prices are positive

Budget exhaustion also holds

For consumer A,

$$4 \cdot \frac{29}{4} + 11 \cdot \frac{29}{11} = 29 + 29 = 58$$

$$4 \cdot 9 + 11 \cdot 2 = 36 + 22 = 58$$

For consumer B,

$$4 \cdot \frac{13}{4} + 11 \cdot \frac{26}{11} = 13 + 26 = 39$$

$$4 \cdot \frac{3}{2} + 11 \cdot 3 = 6 + 33 = 39$$

Hence, the allocation found in part (1), together with the relative price $\frac{p_x}{p_y} = \frac{4}{11}$, satisfies all the conditions of a Walrasian equilibrium

3. The total endowment in the economy is

$$\bar{x} = 9 + \frac{3}{2} = \frac{21}{2} \qquad \bar{y} = 2 + 3 = 5$$

To find the contract curve, we characterize the Pareto efficient allocations
At an interior efficient allocation, the marginal rates of substitution must be equal
For consumer A,

$$u_A = (x_A y_A)^2$$

$$MU_{x_A} = 2x_A y_A^2 \qquad MU_{y_A} = 2x_A^2 y_A$$

Therefore,

$$MRS_A = \frac{MU_{x_A}}{MU_{y_A}} = \frac{y_A}{x_A}$$

For consumer B,

$$u_B = \ln(x_B) + 2\ln(y_B)$$

$$MU_{x_B} = \frac{1}{x_B} \qquad MU_{y_B} = \frac{2}{y_B}$$

Therefore,

$$MRS_B = \frac{MU_{x_B}}{MU_{y_B}} = \frac{y_B}{2x_B}$$

Efficiency requires

$$\frac{y_A}{x_A} = \frac{y_B}{2x_B}$$

Using feasibility,

$$x_B = \frac{21}{2} - x_A \qquad y_B = 5 - y_A$$

we obtain

$$\frac{y_A}{x_A} = \frac{5 - y_A}{2\left(\frac{21}{2} - x_A\right)}$$

$$\frac{y_A}{x_A} = \frac{5 - y_A}{21 - 2x_A}$$

$$y_A(21 - 2x_A) = x_A(5 - y_A)$$

$$21y_A - 2x_A y_A = 5x_A - x_A y_A$$

$$21y_A - x_A y_A = 5x_A$$

$$y_A(21 - x_A) = 5x_A$$

Thus, the contract curve is

$$y_A = \frac{5x_A}{21 - x_A} \qquad 0 < x_A < \frac{21}{2}$$

Equivalently, in terms of both consumers' allocations,

$$(x_A, y_A) = \left(x_A, \frac{5x_A}{21 - x_A}\right)$$

$$(x_B, y_B) = \left(\frac{21}{2} - x_A,\ 5 - \frac{5x_A}{21 - x_A}\right) \qquad 0 < x_A < \frac{21}{2}$$

Now we derive the Utility Possibility Frontier by evaluating utilities along the contract curve
For consumer A,

$$U_A = (x_A y_A)^2$$

$$U_A = \left(x_A \cdot \frac{5x_A}{21 - x_A}\right)^2$$

$$U_A = \left(\frac{5x_A^2}{21 - x_A}\right)^2$$

For consumer B,

$$U_B = \ln(x_B) + 2\ln(y_B)$$

$$U_B = \ln\left(\frac{21}{2} - x_A\right) + 2\ln\left(5 - \frac{5x_A}{21 - x_A}\right)$$

Since

$$5 - \frac{5x_A}{21 - x_A} = \frac{5(21 - 2x_A)}{21 - x_A}$$

we get

$$U_B = \ln\left(\frac{21}{2} - x_A\right) + 2\ln\left(\frac{5(21 - 2x_A)}{21 - x_A}\right)$$

Hence, a parametric representation of the Utility Possibility Frontier is

$$U_A = \left(\frac{5x_A^2}{21 - x_A}\right)^2$$

$$U_B = \ln\left(\frac{21}{2} - x_A\right) + 2\ln\left(\frac{5(21 - 2x_A)}{21 - x_A}\right) \qquad 0 < x_A < \frac{21}{2}$$

Therefore, the contract curve is given by $y_A = \frac{5x_A}{21 - x_A}$, and the UPF is the set of utility pairs generated by this curve

4. From part (1), the equilibrium allocation is

$$(x_A, y_A) = \left(\frac{29}{4}, \frac{29}{11}\right) \qquad (x_B, y_B) = \left(\frac{13}{4}, \frac{26}{11}\right)$$

From part (3), the contract curve is given by

$$y_A = \frac{5x_A}{21 - x_A}$$

We now check whether the equilibrium allocation satisfies this condition. Substituting $x_A = \frac{29}{4}$,

$$y_A = \frac{5 \cdot \frac{29}{4}}{21 - \frac{29}{4}}$$

$$y_A = \frac{\frac{145}{4}}{\frac{84}{4} - \frac{29}{4}}$$

$$y_A = \frac{\frac{145}{4}}{\frac{55}{4}}$$

$$y_A = \frac{145}{55}$$

$$y_A = \frac{29}{11}$$

which is exactly the equilibrium value found in part (1)

Therefore, the allocation found in part (1) lies on the contract curve

Since every feasible allocation on the contract curve is Pareto efficient, it follows that the equilibrium allocation is Pareto efficient

We can also verify this through the marginal rates of substitution

At the equilibrium allocation,

$$MRS_A = \frac{y_A}{x_A} = \frac{\frac{29}{11}}{\frac{29}{4}} = \frac{4}{11}$$

$$MRS_B = \frac{y_B}{2x_B} = \frac{\frac{26}{11}}{2 \cdot \frac{13}{4}} = \frac{\frac{26}{11}}{\frac{13}{2}} = \frac{4}{11}$$

Thus,

$$MRS_A = MRS_B = \frac{4}{11}$$

which confirms again that the equilibrium allocation is efficient

Hence, the equilibrium found in part (1) is Pareto efficient because it lies on the contract curve, and equivalently because the two consumers' marginal rates of substitution are equal at that allocation